

Small-sample likelihood inference in extreme-value regression models

Silvia L.P. Ferrari Eliane C. Pinheiro

Departamento de Estatística, Universidade de São Paulo, Brazil

March 2, 2013

Abstract

We deal with a general class of extreme-value regression models introduced by Barreto-Souza and Vasconcelos (2011). Our goal is to derive an adjusted likelihood ratio statistic that is approximately distributed as χ^2 with a high degree of accuracy. Although the adjusted statistic requires more computational effort than its unadjusted counterpart, it is shown that the adjustment term has a simple compact form that can be easily implemented in standard statistical software. Further, we compare the finite sample performance of the three classical tests (likelihood ratio, Wald, and score), the gradient test that has been recently proposed by Terrell (2002), and the adjusted likelihood ratio test obtained in this paper. Our simulations favor the latter. Applications of our results are presented.

Key words: Extreme-value regression; Gradient test; Gumbel distribution; Likelihood ratio test; Nonlinear models; Score test; Small-sample adjustments; Wald test.

1 Introduction

The extreme-value distributions are frequently used to model extreme events, such as extreme floods and wind speed, and in survival or reliability analysis to model the logarithm of lifetime data. The literature on statistics of extremes has grown fast due to the increasing interest in statistically modeling extreme values (minimum or maximum) in a wide range of areas, such as climatology, hydrology, reliability, finance, insurance, and environmental sciences. A classical reference on the subject is the book by Gumbel (1958). More recent references include Kotz and Nadarajah (2000), Coles (2001), and Castillo et al. (2005), among others.

In this paper, we deal with a general class of extreme-value regression models introduced by Barreto-Souza and Vasconcellos (2011). The authors presented large-sample inference on the parameters, and also considered the issue of correcting the bias of the maximum likelihood estimators in small samples. The study of small-sample inference in extreme-value models

is relevant since the amount of extreme data available for analysis may be small in practical applications and, as Mueller and Rufibach (2009) pointed out, there are very few articles that focus on small sample problems in extreme value theory. Here, we focus on statistical tests in the general class of extreme-value regression models proposed by Barreto-Souza and Vasconcellos (2011) when the sample size is small. Specifically, our goal is to derive Skovgaard's adjusted likelihood ratio statistics in this class of models. We show that the adjustment term has a simple compact form that can be easily implemented from standard statistical software. The adjusted statistic is approximately distributed as χ^2 with a high degree of accuracy. Further, we compare the finite sample performance of the three classical tests (likelihood ratio, Wald, and score), the gradient test that has been recently proposed by Terrell (2002), and the adjusted likelihood ratio test obtained in this paper.

Let y_1, \dots, y_n be independent random variables, where each y_t , $t = 1, \dots, n$, has an extreme-value distribution with parameters μ_t and ϕ_t and density function

$$f(y; \mu_t, \phi_t) = \frac{1}{\phi_t} \exp\left(-\frac{y - \mu_t}{\phi_t}\right) \exp\left\{-\exp\left(-\frac{y - \mu_t}{\phi_t}\right)\right\}, \quad y \in \mathbb{R}, \quad (1)$$

where $\mu \in \mathbb{R}$ and $\phi > 0$ are the location and dispersion parameters, respectively. The mean and the variance of y_t are $E(y_t) = \mu_t + \mathcal{E}\phi_t$ and $\text{var}(y_t) = \phi_t^2\pi^2/6$, respectively, where \mathcal{E} is the Euler constant; $\mathcal{E} \approx 0.5772$. If y has an extreme-value distribution with parameters μ and ϕ , we write $y \sim EV_{\max}(\mu, \phi)$. This distribution is also called Gumbel or type I extreme-value distribution. Here, we call it the maximum extreme-value distribution to contrast it with the minimum extreme-value distribution (see Section 3 below), which is also called Gumbel or type I extreme-value in the statistical literature.

The maximum extreme-value regression model with dispersion covariates is defined by (1) and by two systematic components given by

$$g(\mu_t) = \eta_t = \eta(x_t, \beta) \quad (2)$$

and

$$h(\phi_t) = \delta_t = \delta(z_t, \gamma), \quad (3)$$

where $\beta = (\beta_1, \dots, \beta_k)^\top$ and $\gamma = (\gamma_1, \dots, \gamma_m)^\top$ are vectors of unknown regression parameters ($\beta \in \mathbb{R}^k$ and $\gamma \in \mathbb{R}^m$, $k + m < n$) and x_t and z_t are observations on covariates. Here, $\eta(\cdot, \cdot)$ and $\delta(\cdot, \cdot)$ are continuously twice differentiable (possibly nonlinear) functions in the second argument. Finally, $g(\cdot)$ and $h(\cdot)$ are known strictly monotonic and twice differentiable link functions that map \mathbb{R} and \mathbb{R}^+ into \mathbb{R} , respectively. Let X be the derivative matrix of $\eta = (\eta_1, \dots, \eta_n)^\top$ with respect to β^\top . Analogously, let Z be the derivative matrix of $\delta = (\delta_1, \dots, \delta_n)^\top$ with respect to γ^\top . It is assumed that $\text{rank}(X) = k$ and $\text{rank}(Z) = m$ for all β and γ .

The paper unfolds as follows. In Section 2, we derive Skovgaard's adjusted likelihood ratio statistic for testing hypothesis on the parameters of the model definide by (1)-(3). In Section 3,

we extend our results to a general class of minimum extreme-value regression models. Section 4 is devoted to a simulation study to compare the performance of the three classical tests, the gradient test, and the adjusted likelihood ratio test. Our simulation results clearly favor the adjusted test proposed in this paper. In Section 5, we illustrate the use of our results in two real data sets. Section 6 closes the paper with a discussion. Technical details are left for an appendix.

2 Main results

Let $\ell(\theta)$ be the log-likelihood function of the model defined by (1)-(3) given the vector of observations $y = (y_1, \dots, y_n)$. We have

$$\ell(\theta) = \sum_{t=1}^n \ell_t(\mu_t, \phi_t),$$

where

$$\ell_t(\mu_t, \phi_t) = -\log(\phi_t) - \frac{y_t - \mu_t}{\phi_t} - \exp\left(-\frac{y_t - \mu_t}{\phi_t}\right),$$

with μ_t and ϕ_t defined so that (2) and (3) hold. In matrix notation, the log-likelihood function can be written as

$$\ell(\theta) = [-l^\top - \mathbf{z}^\top - \check{\mathbf{z}}^\top] \mathbf{1}, \quad (4)$$

where $l = (\ln \phi_1, \dots, \ln \phi_n)^\top$, $\mathbf{z} = (\mathbf{z}_1, \dots, \mathbf{z}_n)^\top$, and $\check{\mathbf{z}} = (\exp(-\mathbf{z}_1), \dots, \exp(-\mathbf{z}_n))^\top$, with $\mathbf{z}_t = (y_t - \mu_t)/\phi_t$, and $\mathbf{1}$ is the n -dimensional column vector of ones. The score function, obtained by differentiating the log-likelihood function with respect to the unknown parameters, is denoted by $U \equiv (U_\beta(\beta, \gamma)^\top, U_\gamma(\beta, \gamma)^\top)^\top$, with I and J denoting the expected and observed information matrices. We have

$$U_\beta(\beta, \gamma) = X^\top \Phi^{-1} T(\mathbf{1} - \check{\mathbf{z}}), \quad (5)$$

$$U_\gamma(\beta, \gamma) = Z^\top \Phi^{-1} H(\mathbf{z} - \check{Z}\check{\mathbf{z}} - \mathbf{1}), \quad (6)$$

$$J = \begin{bmatrix} J_{\beta\beta} & J_{\beta\gamma} \\ J_{\gamma\beta} & J_{\gamma\gamma} \end{bmatrix}, \quad I = \begin{bmatrix} I_{\beta\beta} & I_{\beta\gamma} \\ I_{\gamma\beta} & I_{\gamma\gamma} \end{bmatrix},$$

with

$$\begin{aligned} J_{\beta\beta} &= X^\top \Phi^{-1} T(\check{Z}\Phi^{-1} + (\mathcal{I} - \check{Z})ST)TX - [\mathbf{1}^\top (\mathcal{I} - \check{Z})T\Phi^{-1}] [\dot{X}], \\ J_{\beta\gamma} &= J_{\gamma\beta}^\top = X^\top \Phi^{-1} T(\mathcal{I} - \check{Z} + Z\check{Z})H\Phi^{-1}Z, \\ J_{\gamma\gamma} &= Z^\top \Phi^{-1} H((- \mathcal{I} + 2Z - 2Z\check{Z} + Z^2\check{Z})\Phi^{-1} \\ &\quad + (-\mathcal{I} + Z - Z\check{Z})QH)HZ + [\mathbf{1}^\top (\mathcal{I} - Z + Z\check{Z})H\Phi^{-1}] [\dot{Z}], \\ I_{\beta\beta} &= X^\top \Phi^{-2} T^2 X, \quad I_{\beta\gamma} = I_{\gamma\beta}^\top = (\mathcal{E} - 1)X^\top \Phi^{-1} TH\Phi^{-1}Z, \\ I_{\gamma\gamma} &= (1 + \Gamma^{(2)}(2))Z^\top \Phi^{-1} H^2 \Phi^{-1} Z, \end{aligned}$$

where \mathcal{I} is an $n \times n$ identity matrix, $\mathcal{Z} = \text{diag}\{\mathfrak{z}_1, \dots, \mathfrak{z}_n\}$, $\check{\mathcal{Z}} = \text{diag}\{\exp(-\mathfrak{z}_1), \dots, \exp(-\mathfrak{z}_n)\}$, $\Phi = \text{diag}\{\phi_1, \dots, \phi_n\}$, $T = \text{diag}\{1/g'(\mu_1), \dots, 1/g'(\mu_n)\}$, $H = \text{diag}\{1/h'(\phi_1), \dots, 1/h'(\phi_n)\}$, $S = \text{diag}\{g''(\mu_1), \dots, g''(\mu_n)\}$, $Q = \text{diag}\{h''(\phi_1), \dots, h''(\phi_n)\}$, $\dot{X} = \partial^2 \eta / \partial \beta \partial \beta^\top$ and $\dot{Z} = \partial^2 \delta / \partial \gamma \partial \gamma^\top$ are $n \times k \times k$ and $n \times m \times m$ arrays, respectively, and $\Gamma(\cdot)$ denotes the gamma function, with $\Gamma^{(1)}(\cdot)$ and $\Gamma^{(2)}(\cdot)$ being its first and second derivatives, respectively. The other quantities are as before.

Let $\theta = (\beta^\top, \gamma^\top)^\top$ be the unknown parameter vector that indexes the extreme-value regression model (1)-(3). In what follows, $\nu = (\nu_1, \dots, \nu_r)^\top$ represents the parameter of interest and $\psi = (\psi_1, \dots, \psi_s)^\top$ is the nuisance parameter; note that $r + s = k + m$. We consider likelihood-based tests of the null hypothesis $\mathcal{H}_0 : \nu = \nu_0$, where ν_0 is a fixed r -vector. Clearly, such tests may be inverted to give confidence sets for ν . Further, let $J_{\psi\psi}$ denote the $s \times s$ observed information matrix corresponding to ψ . Similarly, $A_{\psi\psi}$ denotes a matrix formed from the $(r + s) \times (r + s)$ matrix A by dropping the rows and columns that correspond to the interest parameter. Additionally, hat and tilde indicate evaluation at the unrestricted ($\hat{\theta}$) and at the restricted ($\tilde{\theta}$) maximum likelihood estimator of θ under \mathcal{H}_0 , respectively. For instance, $\hat{I} = I(\hat{\theta})$, $\tilde{I} = I(\tilde{\theta})$, and $\hat{J} = J(\hat{\theta})$.

Skovgaard (2001) derived an adjusted likelihood ratio statistic given by

$$w^* = w - 2 \log \zeta, \quad (7)$$

where $w = 2(\ell(\hat{\theta}) - \ell(\tilde{\theta}))$ is the likelihood ratio statistic,

$$\zeta = \frac{\{|\tilde{I}| |\hat{I}| |\tilde{J}_{\psi\psi}|\}^{1/2}}{|\tilde{\Upsilon}| |\{\tilde{I}\tilde{\Upsilon}^{-1}\hat{I}\tilde{\Upsilon}^{-1}\tilde{J}_{\psi\psi}\}^{1/2}} \frac{\{\tilde{U}^\top \tilde{\Upsilon}^{-1} \hat{I} \hat{J}^{-1} \tilde{\Upsilon}^{-1} \tilde{U}\}^{r/2}}{w^{r/2-1} \tilde{U}^\top \tilde{\Upsilon}^{-1} \bar{q}},$$

and \bar{q} and $\tilde{\Upsilon}$ come, respectively, from

$$q = E_{\theta_1}[U(\theta_1) (\ell(\theta_1) - \ell(\theta))] \quad (8)$$

and

$$\Upsilon = E_{\theta_1}[U(\theta_1) U^\top(\theta)] \quad (9)$$

by inserting $\hat{\theta}$ for θ_1 and $\tilde{\theta}$ for θ after the expected values are computed. Note that \bar{q} is an $(r + s)$ -vector and $\tilde{\Upsilon}$ is an $(r + s) \times (r + s)$ matrix. Under \mathcal{H}_0 , w is distributed as χ_r^2 with error of order n^{-1} while w^* follows this distribution with high degree of accuracy (Skovgaard, 2001, p. 7). Simulation results in Ferrari and Pinheiro (2010) and Ferrari and Cysneiros (2008) suggest that tests that use w^* are much less size distorted than those that are based on w .

In order to obtain the adjusted likelihood ratio statistic (7) in the extreme-value regression model (1)-(3), one needs to obtain the score vector, the observed and expected information matrices, J and I , respectively, the vector q , and the matrix Υ . We obtained

$$\bar{q} = \begin{bmatrix} \hat{X}^\top \hat{\Phi}^{-1} \hat{T} C (\mathcal{I} - M \check{D}) \mathbf{1} \\ \hat{Z}^\top \hat{\Phi}^{-1} \hat{H} \{C(\mathcal{E} \mathcal{I} + N \check{D}) - \mathcal{I}\} \mathbf{1} \end{bmatrix}$$

and

$$\overline{\Upsilon} = \begin{bmatrix} \hat{X}^\top \hat{\Phi}^{-1} \hat{T} C M \check{D} \tilde{T} \tilde{\Phi}^{-1} \tilde{X} & \hat{X}^\top \hat{\Phi}^{-1} \hat{T} C \{\mathcal{I} + \check{D}(MD - M - CN)\} \tilde{H} \tilde{\Phi}^{-1} \tilde{Z} \\ -\hat{Z}^\top \hat{\Phi}^{-1} \hat{H} C N \check{D} \tilde{T} \tilde{\Phi}^{-1} \tilde{X} & \hat{Z}^\top \hat{\Phi}^{-1} \hat{H} C \{\mathcal{E}\mathcal{I} + \check{D}(N + CP - ND)\} \tilde{H} \tilde{\Phi}^{-1} \tilde{Z} \end{bmatrix},$$

where $C = \text{diag}\{\phi_{11}/\phi_1, \dots, \phi_{1n}/\phi_n\}$, $D = \text{diag}\{(\mu_{11} - \mu_1)/\phi_1, \dots, (\mu_{1n} - \mu_n)/\phi_n\}$, $\check{D} = \text{diag}\{\exp(-(\mu_{11} - \mu_1)/\phi_1), \dots, \exp(-(\mu_{1n} - \mu_n)/\phi_n)\}$, $M = \text{diag}\{\Gamma(1 + \phi_{11}/\phi_1), \dots, \Gamma(1 + \phi_{1n}/\phi_n)\}$, $N = \text{diag}\{\Gamma^{(1)}(1 + \phi_{11}/\phi_1), \dots, \Gamma^{(1)}(1 + \phi_{1n}/\phi_n)\}$, $P = \text{diag}\{\Gamma^{(2)}(1 + \phi_{11}/\phi_1), \dots, \Gamma^{(2)}(1 + \phi_{1n}/\phi_n)\}$, and the other quantities are as given above. Details of the derivations of \bar{q} and $\overline{\Upsilon}$ are given in the Appendix.

3 Minimum extreme-value regression model

Let y_1, \dots, y_n be independent random variables, where each y_t , $t = 1, \dots, n$, has a minimum extreme-value distribution with parameters μ_t and ϕ_t and density function

$$f(y; \mu_t, \phi_t) = \frac{1}{\phi_t} \exp\left(\frac{y - \mu_t}{\phi_t}\right) \exp\left\{-\exp\left(\frac{y - \mu_t}{\phi_t}\right)\right\}, \quad y \in \mathbb{R}, \quad (10)$$

where $\mu \in \mathbb{R}$ and $\phi > 0$ are the location and dispersion parameters, respectively. The mean and the variance of y_t are $E(y_t) = \mu_t - \mathcal{E}\phi_t$ and $\text{var}(y_t) = \phi_t^2 \pi^2/6$, respectively. If y has a minimum extreme-value distribution we write $y \sim EV_{\min}(\mu, \phi)$. A useful property of the minimum extreme-value distribution is as follows:

$$y \sim EV_{\min}(\mu, \phi) \implies -y \sim EV_{\max}(-\mu, \phi). \quad (11)$$

The minimum extreme-value regression model with dispersion covariates is defined by (10) and by the systematic components (2) and (3).

From (11), it is easy to see that the minimum extreme-value regression model (10) with systematic components (2) and (3) is equivalent to the (maximum) extreme-value regression model (1) for the response variables $v_1 = -y_1, \dots, v_n = -y_n$ with systematic components $g^*(\mu_t) = \mu_t^* = \eta_t^* = -g^{-1}(\eta(x_t, \beta))$ and $h^*(\phi_t) = h(\phi_t) = \delta(z_t, \gamma)$. Hence, inference for the minimum extreme-value regression model (10) with systematic components (2) and (3) can be performed from the results derived in Section 2 by changing the signs of the observations on the response variable and using an identity link function for the location parameter with the modified predictor η_t^* . As a result, the adjusted likelihood ratio statistic derived in Section 2 can be easily computed for the minimum extreme-value regression model.

4 Monte Carlo simulation results

We now present Monte Carlo simulation results on the small sample behaviour of the likelihood ratio test (w), the Wald test (W), the score test (S_R), the gradient test (S_T), and the adjusted

likelihood ratio test (w^*). The Wald, score, and gradient statistics are given by $W = (\hat{\nu} - \nu_0)^\top (\hat{I}^{\nu\nu})^{-1} (\hat{\nu} - \nu_0)$, $S_R = \tilde{U}_\nu^\top \tilde{I}^{\nu\nu} \tilde{U}_\nu$, and $S_T = \tilde{U}_\nu^\top (\hat{\nu} - \nu_0)$. Note that the gradient statistic is very simple to compute since it does not involve the information matrix, neither the observed one nor the expected one.

The maximum likelihood estimation of the unknown parameters was performed using the quasi-Newton BFGS nonlinear optimization algorithm with analytical derivatives developed by Broyden, Fletcher, Goldfarf & Shanno (see, for instance, Press et al. (1992)) and implemented in the function MAXBFGS in the matrix language programming `0x` (Doornik, 2009).

All the size simulation results are based on 10,000 Monte Carlo replications and the nominal level of the tests are $\alpha = 10\%$, 5% , and 1% . We also present power simulation results. Since the different tests display different sizes when a χ^2 distribution is used, we simulated 500,000 samples to estimate the critical values of the tests that give exact size, i.e., size equal to the chosen nominal level. Our power simulation results are obtained using exact critical values.

We consider model (1) with constant dispersion and location parameter given by

$$\mu_t = \beta_1 + \beta_2 x_{t2} + \beta_3 x_{t3} + \beta_4 x_{t4} + \beta_5 x_{t5},$$

which we refer as ‘model 1’. Three null hypotheses are considered, $\mathcal{H}_0 : \beta_2 = 0$ ($r = 1$), $\mathcal{H}_0 : \beta_2 = \beta_3 = 0$ ($r = 2$), and $\mathcal{H}_0 : \beta_2 = \beta_3 = \beta_4 = 0$ ($r = 3$), and these are to be tested against a two-sided alternative. For the first case, we set $\beta_1 = 1$, $\beta_2 = 0$, $\beta_3 = 1$, $\beta_4 = 6$, and $\beta_5 = -3$; for the second case, $\beta_1 = 1$, $\beta_2 = \beta_3 = 0$, $\beta_4 = 6$, and $\beta_5 = -3$; and for the third case, $\beta_1 = 1$, $\beta_2 = \beta_3 = \beta_4 = 0$, and $\beta_5 = -3$. The value of ϕ was fixed at $\phi = 0.1$.¹ The covariate values were obtained as random draws from a $\mathcal{U}(-0.5, 0.5)$ distribution and the sample sizes are 15, 20, 30, and 40.

Table 1 presents the null rejection rates of the five tests. It can be noticed that the likelihood ratio and Wald tests are markedly liberal in small samples. For instance, for $n = 15$, $r = 1$, and $\alpha = 5\%$, the null rejection rates of these tests are 11.6% and 20.8%, respectively. The gradient test is liberal in many cases but not as much as the likelihood ratio and the Wald tests. The score test is less liberal and displays conservative behavior in some cases. The adjusted likelihood ratio test is clearly the least size distorted. For the case mentioned above, the null rejection rates of the score, gradient, and adjusted likelihood ratio tests are 5.8%, 9.0%, and 5.0%, respectively.

Figure 1 shows the plots of the relative quantile discrepancies versus corresponding asymptotic quantiles for $r = 1, 2, 3$, and $n = 20, 30, 40$. Relative quantile discrepancy is defined as the difference between exact (estimated by simulation) and asymptotic quantiles divided by the latter. The closer to zero the relative quantile discrepancy, the better is the approximation of the exact null distribution of the test statistic by the limiting χ^2 distribution. The plots confirm

¹For linear extreme-value regression models with constant dispersion, the null distributions of the five statistics do not depend on ϕ . The proof is omitted to save space.

Table 1: Null rejection rates (%); model 1

r	n	$\alpha = 10\%$					$\alpha = 5\%$					$\alpha = 1\%$				
		w	W	S_R	S_T	w^*	w	W	S_R	S_T	w^*	w	W	S_R	S_T	w^*
1	15	19.3	28.2	11.5	16.9	10.4	11.6	20.8	5.8	9.0	5.0	4.0	11.1	1.1	1.8	1.0
	20	16.8	22.1	11.3	15.0	9.9	10.0	15.6	5.4	7.8	4.8	3.0	7.0	0.9	1.5	0.9
	30	14.9	18.9	10.9	13.8	10.6	8.5	12.0	5.3	7.1	5.4	2.4	4.6	0.9	1.4	1.2
	40	13.0	15.7	10.3	12.1	10.1	7.0	9.5	4.9	6.2	5.1	2.0	3.5	0.9	1.3	1.1
2	15	22.9	36.5	12.7	16.4	7.7	14.3	28.7	6.2	7.9	3.5	4.8	17.1	1.1	0.8	0.6
	20	18.8	28.8	11.6	14.1	9.2	11.4	20.9	5.4	7.2	4.6	3.8	10.5	0.9	0.8	0.8
	30	16.0	23.4	10.8	13.0	10.2	9.2	15.7	5.2	6.6	5.2	2.3	6.7	0.8	0.9	0.9
	40	13.9	19.0	10.2	11.7	9.8	7.6	12.0	5.0	6.1	5.2	1.9	4.5	0.7	0.9	0.9
3	15	23.4	42.9	9.1	12.6	6.8	14.9	34.2	3.4	4.8	3.0	5.1	21.2	0.2	0.2	0.5
	20	19.8	33.8	9.8	12.3	8.7	12.1	25.1	4.4	5.4	4.4	3.8	14.0	0.5	0.4	0.8
	30	16.6	26.4	10.1	12.1	9.9	9.5	18.2	4.7	5.4	4.9	2.5	8.6	0.6	0.6	0.9
	40	14.7	21.4	10.2	11.3	10.3	8.3	14.0	4.6	5.5	5.2	2.1	5.8	0.7	0.7	0.9

the tendency of the likelihood ratio and the Wald tests of rejecting the null hypothesis with higher frequency than expected based on the nominal level. It is clear that the distribution of the adjusted likelihood ratio statistic (w^*) closely agrees with the reference distribution. The effect of the proposed adjustment becomes evident. Notice that the best agreement between the exact distribution and its asymptotic counterpart is achieved by w^* for all the sample sizes.

We now focus on the power comparisons of the five tests for $r = 1$, $n = 30$, and $\alpha = 10\%$. The rejection rates were obtained under the alternative hypothesis $\mathcal{H}_1 : \beta_2 = \epsilon$ for different values of ϵ through Monte Carlo simulation. Figure 2 gives the plots of the power function of the tests. Visual inspection shows that the curves are practically coincident, i.e., the five tests display similar powers.

Now, consider model (1) with systematic components for the location and scale parameters, respectively, given by

$$\mu_t = \beta_1 + \beta_2 x_{t2} + \beta_3 x_{t3} + \beta_4 x_{t4},$$

and

$$\ln(\phi_t) = \gamma_1 + \gamma_2 z_{t2} + \gamma_3 z_{t3} + \gamma_4 z_{t4},$$

wich we refer to as ‘model 2’.

We consider three different null hypotheses, $\mathcal{H}_0 : \beta_4 = 0$ ($r = 1$), $\mathcal{H}_0 : \beta_3 = \beta_4 = 0$ ($r = 2$), and $\mathcal{H}_0 : \beta_2 = \beta_3 = \beta_4 = 0$ ($r = 3$), and these are to be tested against two-sided alternatives. The values for the β s are $\beta_1 = 1$, $\beta_2 = 1$, $\beta_3 = 6$, and $\beta_4 = 0$; $\beta_1 = 1$, $\beta_2 = 1$, and $\beta_3 = \beta_4 = 0$; and $\beta_1 = 1$ and $\beta_2 = \beta_3 = \beta_4 = 0$, for the first, second, and third cases, respectively. Further, we set $\gamma_1 = \ln(0.1) \approx -2.30$, $\gamma_2 = -2$, $\gamma_3 = -2$, and $\gamma_4 = 0.1$. The covariate values were randomly drawn from a $\mathcal{U}(-0.5, 0.5)$ distribution and the sample sizes are 40, 50, 60, and 70.

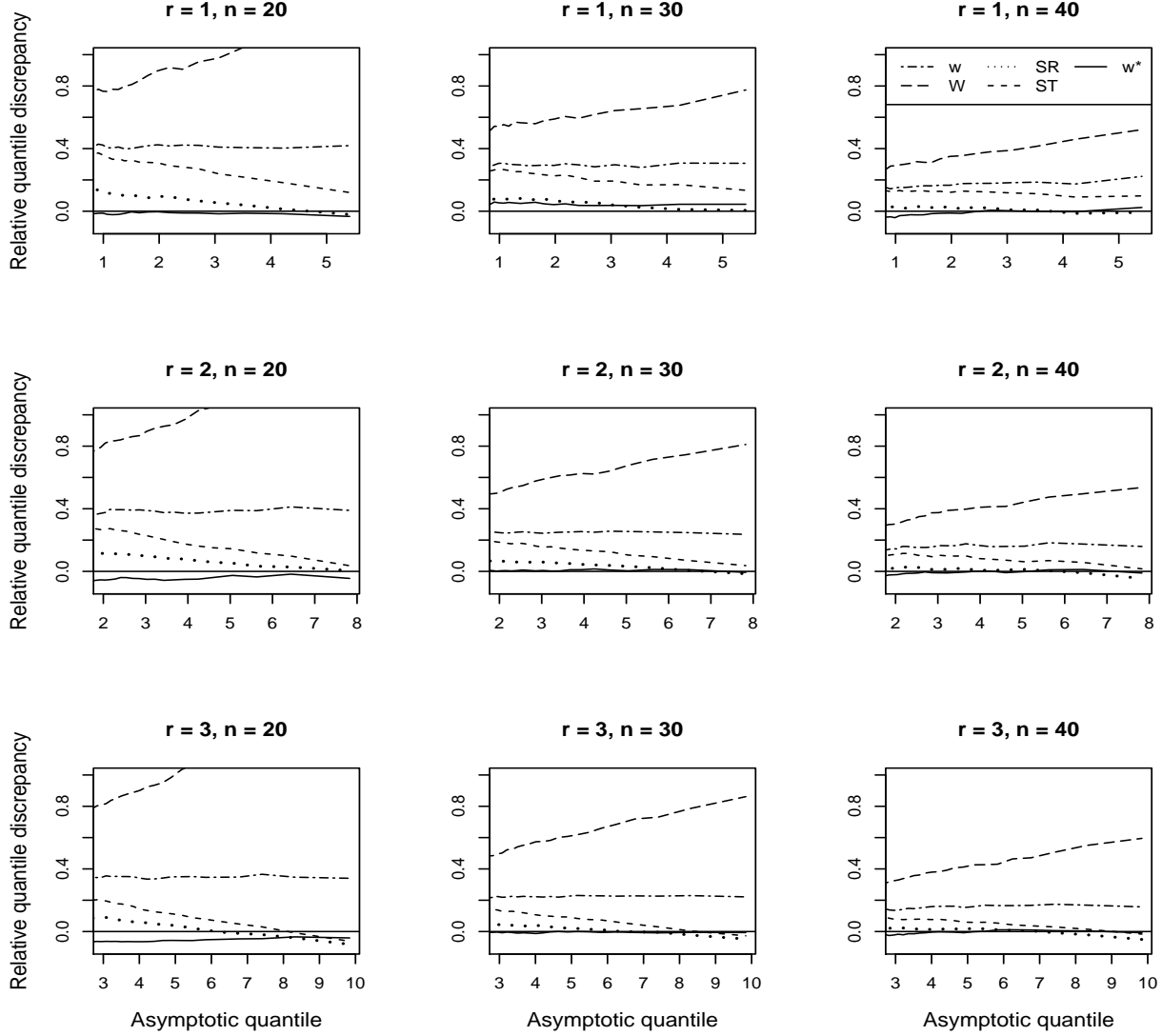


Figure 1: Relative quantile discrepancies, model 1.

Table 2 gives the null rejection rates of the five tests and Figure 3 shows the plots of relative quantile discrepancies. We note that the results for model 2 show similarity with those for model 1. The likelihood ratio and Wald tests are clearly oversized, i.e. its type I error probability is greater than the nominal level, and their distributions are much different from the asymptotic χ^2 distribution if the sample is not large. Again, the best agreement between the true and asymptotic quantiles is reached by w^* , the adjusted likelihood ratio statistic proposed in this paper. The score test presents good behavior but tends to be conservative when $r > 1$. The gradient test is liberal in many cases, but high order quantiles of the gradient statistic are close to the asymptotic quantiles. We also performed power simulation comparisons among the five tests. Overall, the tests are equally powerful when true critical values are used.

We now consider model (1) with constant dispersion and a nonlinear specification for the

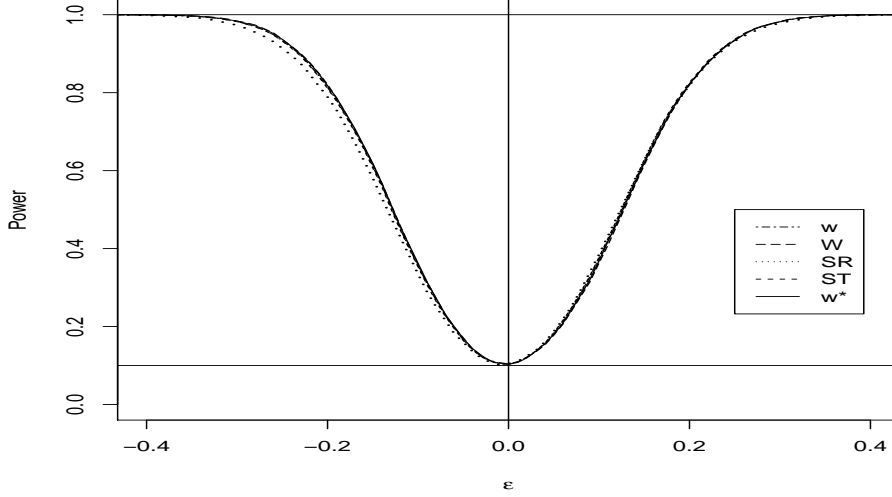


Figure 2: Power of the tests; model 1; $r = 1$, $n = 30$, $\alpha = 10\%$.

location parameter given by

$$\mu_t = \beta_0 + \beta_1 x_{t1} + x_{t2}^{\beta_2},$$

wich we refer to as ‘model 3’.

Here, X is an $n \times 3$ matrix whose t -th row is $(1, x_{t1}, \ln(x_{t2})x_{t2}^{\beta_2})$, $Z = \mathcal{I}$, $T = \mathcal{I}$, $H = \mathcal{I}$, S and Q are matrices of zeros, $\dot{X} = \partial^2 \eta / \partial \beta \partial \beta^\top$ is such that $\partial^2 \eta_t / \partial \beta_2 \partial \beta_2 = \ln(x_{t2})^2 x_{t2}^{\beta_2}$ for $t = 1, \dots, n$ and zero otherwise, and $\dot{Z} = \partial^2 \delta / \partial \gamma \partial \gamma^\top$ is an $n \times m \times m$ array of zeros. The bracket product of the $1 \times n$ vector $\left[\mathbf{1}^\top (\mathcal{I} - \check{Z}) T \Phi^{-1} \right]$ and the $n \times 3 \times 3$ array \dot{X} is an $1 \times 3 \times 3$ array, i.e., a 3×3 matrix, whose (i, j) -th element is

$$\sum_{t=1}^n \left\{ \left(\frac{1}{\phi_t} - \frac{1}{\phi_t} \exp \left(-\frac{y_t - \mu_t}{\phi_t} \right) \right) \frac{1}{g'(\mu_t)} \ln(x_{t2})^2 x_{t2}^{\beta_2} \right\}$$

if $(i, j) = (3, 3)$ and zero otherwise.

The null hypothesis under test is $\mathcal{H}_0 : \beta_2 = 0$ ($r = 1$). We set $\phi = e^{0.1} \approx 1.1$, $\beta_0 = 1$, $\beta_1 = 1$, and $\beta_2 = 0$, and the covariate values were drawn from a $\mathcal{U}(0, 1)$ distribution. The sample sizes are 15, 20, 30, and 40. Table 3 and Figure 4 show our simulation results.

From Table 3, we note that the tests that use w and W are typically liberal while the null rejection of the other tests keeps their sizes closer to the nominal levels. The adjusted likelihood ratio test and the score test display better performance than the others. For example, for $n = 15$ and $\alpha = 10\%$, the null rejection rates of the tests are 16.6% (likelihood ratio), 22.2% (Wald), 13.4% (gradient), 10.2% (score), and 10.0% (adjusted likelihood ratio).

Figure 4 shows that the reference distribution is not a good approximation for the null distribution of w and W , but is close to the true null distribution of the score and the adjusted likelihood ratio statistics. The high order quantiles of the gradient statistic closely agree with

Table 2: Null rejection rates (%); model 2

r	n	$\alpha = 10\%$					$\alpha = 5\%$					$\alpha = 1\%$				
		w	W	S_R	S_T	w^*	w	W	S_R	S_T	w^*	w	W	S_R	S_T	w^*
1	40	17.5	24.2	10.8	15.3	11.1	10.4	16.8	5.7	8.4	6.1	3.4	8.0	1.0	1.7	1.6
	50	15.9	20.6	10.8	14.1	10.1	8.9	13.4	5.2	7.2	5.3	2.3	5.4	0.8	1.4	1.1
	60	15.9	21.9	10.6	13.9	10.7	9.3	15.3	5.1	7.0	5.7	2.4	7.0	0.8	1.2	1.2
	70	14.8	19.9	10.0	12.9	10.5	8.4	13.1	5.0	6.6	5.4	2.0	5.6	0.7	1.1	1.2
2	40	21.7	36.9	8.3	13.7	11.8	13.7	28.6	3.7	6.4	6.6	4.5	16.6	0.6	0.9	2.0
	50	18.6	29.4	9.0	13.5	10.9	11.0	21.4	3.9	6.3	5.7	3.2	11.1	0.5	0.9	1.4
	60	17.0	26.9	9.0	12.8	10.6	9.7	19.6	3.9	5.9	5.4	2.9	9.9	0.5	0.9	1.3
	70	15.8	24.5	8.8	12.4	10.3	9.0	17.1	4.0	5.6	5.2	2.3	8.0	0.6	0.9	1.2
3	40	23.1	43.2	7.9	12.1	11.9	14.1	34.6	3.3	5.1	6.8	4.7	21.7	0.3	0.6	2.1
	50	19.3	34.0	8.1	12.1	10.9	11.5	25.9	3.5	5.2	5.6	3.4	14.4	0.4	0.7	1.4
	60	17.2	30.4	8.1	11.4	10.1	9.9	22.3	3.6	5.4	5.2	2.8	11.6	0.5	0.8	1.3
	70	16.4	27.3	8.3	11.4	10.2	9.4	19.8	3.6	5.0	5.0	2.3	9.2	0.6	0.7	1.1

Table 3: Null rejection rates (%); model 3

n	$\alpha = 10\%$					$\alpha = 5\%$					$\alpha = 1\%$				
	w	W	S_R	S_T	w^*	w	W	S_R	S_T	w^*	w	W	S_R	S_T	w^*
15	16.6	22.2	10.2	13.4	10.0	9.8	15.8	4.6	6.3	5.0	2.9	8.2	0.9	1.1	1.1
20	14.1	19.2	9.5	11.9	9.9	7.7	12.7	4.5	5.7	4.8	2.2	5.8	1.1	1.0	1.1
30	12.5	16.3	9.3	11.1	9.7	6.8	10.3	4.6	5.7	5.1	1.5	3.9	1.3	1.0	0.9
40	12.2	15.1	9.8	11.2	10.3	6.7	9.3	4.9	5.6	5.2	1.5	3.4	1.4	1.1	1.0

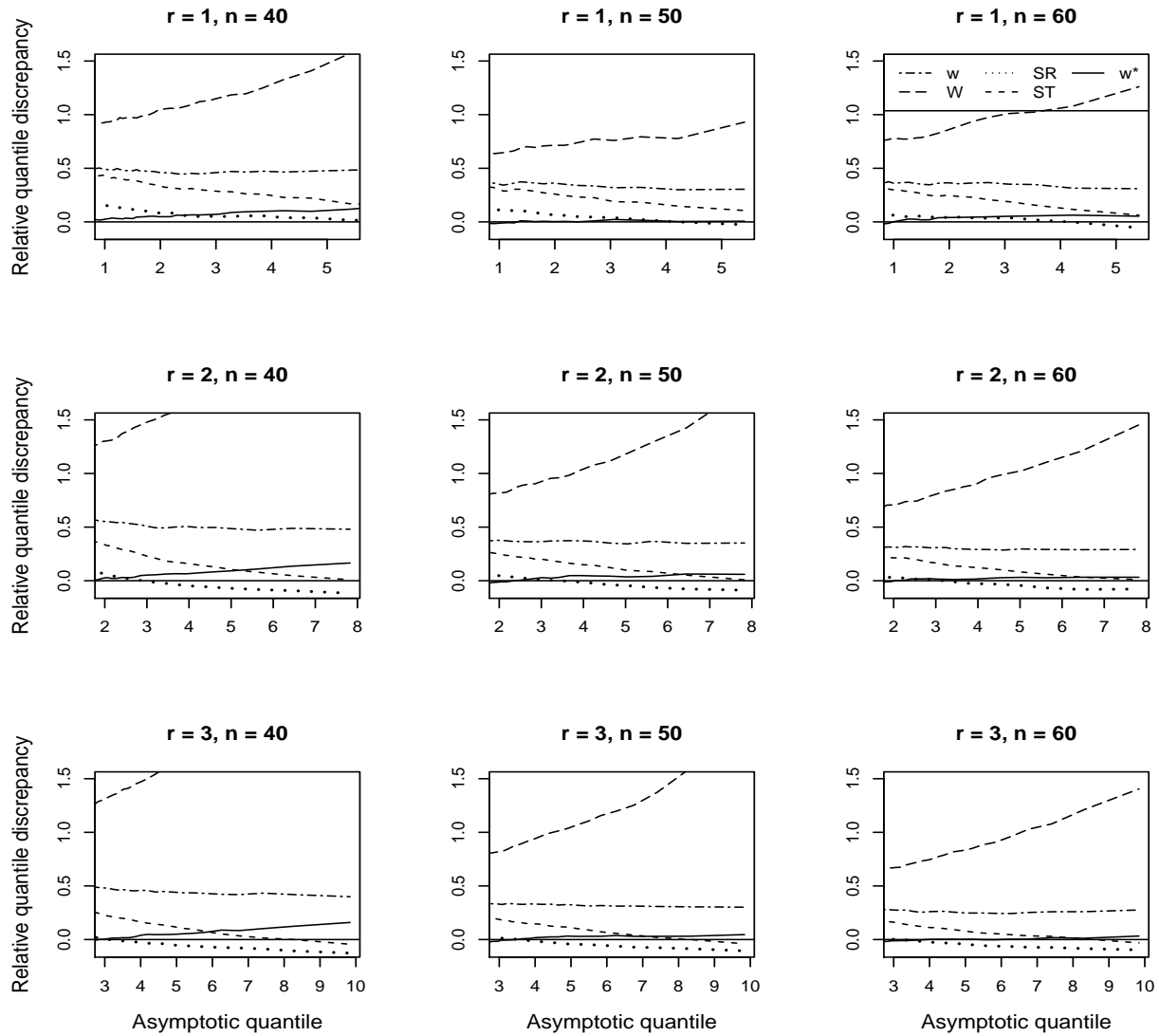


Figure 3: Relative quantile discrepancy; model 2.

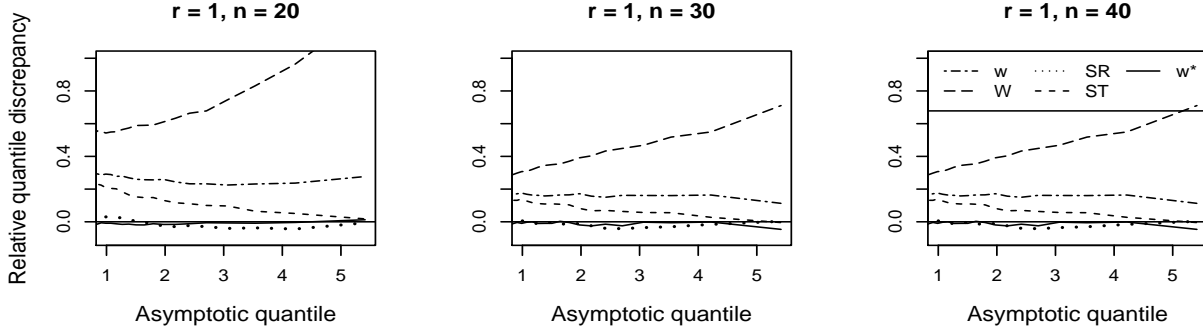


Figure 4: Relative quantile discrepancy, model 3.

the corresponding quantiles of the reference distribution. The five tests have similar power performance (results not shown).

5 Applications

In this section, we illustrate applications of our results in two real datasets. First, we deal with a dataset presented in Faivre & Masle (1988) and in Huet et al. (2004). The aim is to study the growth of winter wheat, by measuring the differences in dry weights of wheat tillers and stems. The explanatory variable x , measured on a cumulative degree-days scale, is an integral in time of all temperatures at which the wheat is submitted that are above the smallest temperature at which wheat can develop. Temperatures are measured in degrees Celsius and time is measured in days, the initial time being determined by the physiological state of the wheat. Plants growing on $n = 18$ randomly chosen small areas of about $0.15 m^2$ are harvested each week and the dry weights of the tillers for plants harvested from each area are measured in milligrams. A detailed description of the data can be found in Huet et al. (2004, p. 61).

Barreto-Souza and Vasconcellos (2011) assumed that the dry weight of tillers (y_1, \dots, y_{18}) is independent and follows a nonlinear extreme-value regression model (1) with

$$\mu_t = \beta_0 + e^{\beta_1 + \beta_2 x_t} \quad \text{and} \quad \ln \phi_t = \gamma_1 x_t, \quad t = 1, \dots, 18.$$

They focused on the issue of correcting the bias of the maximum likelihood estimates (MLEs) of the parameters. The authors constructed confidence intervals based on the asymptotic normality of the MLEs and of the bias-corrected MLEs. Their simulation study suggested that the asymptotic confidence intervals centered at the bias-corrected estimators produce coverage probability closer to the nominal confidence coefficient than those centered in the uncorrected MLEs. However, the choice of the estimator (uncorrected or corrected MLE) does not change the approximation error between the true coverage probability and the nominal confidence coefficient. In fact, the correction that they derived only guarantees that the bias of

the corrected estimators are of order $O(n^{-2})$, but does not change the convergence rate of the distribution of the estimators to the normal distribution.

Here, we illustrate the use of the five test statistics, namely the likelihood ratio, Wald, score, gradient, and adjusted statistic derived in this paper, to obtain interval estimates for the parameters. By choosing the 5% nominal level, the approximate confidence coefficient is 95%. We emphasize that the confidence intervals obtained from the adjusted likelihood ratio statistic have coverage probabilities that are approximately equal to 95% with a high degree of accuracy. This is guaranteed by the theoretical results in Skovgaard (2001) and is confirmed in our simulation study.

The 95% confidence intervals obtained by inverting the five tests are presented in Table 4 and in Figure 5. It can be seen that the intervals obtained from the likelihood ratio and the Wald tests tend to be shorter than those obtained from the other tests as expected, since the first two are the most liberal. We emphasize that theoretical and empirical findings indicate that the confidence intervals constructed from the adjusted likelihood ratio statistic should be regarded as the most accurate.

Table 4: 95% confidence intervals

	w	W	S_R	S_T	w^*
β_0	(47.6; 104.7)	(52.7; 110.5)	(31.4; 100.5)	(38.6 ; 103.7)	(39.7; 104.9)
β_1	(-5.084; -1.111)	(-4.558; -0.922)	(-7.604; -1.200)	(-6.035; -1.113)	(-5.486; -0.956)
β_2	(0.0117; 0.0175)	(0.0114; 0.0167)	(0.0118; 0.0213)	(0.0117; 0.0189)	(0.0115; 0.0181)
γ_1	(0.00610; 0.00723)	(0.00603; 0.00720)	(0.00619; 0.00740)	(0.00612; 0.00727)	(0.00622; 0.00751)

Our second application deals with a dataset consisting of 34 men's decathlon performance at the 1988 Olympic Games Hand et al. (1996, p. 304).² We assume that the score in high jump follows an extreme-value regression model (1) with constant dispersion and systematic component for the location parameter given by

$$\mu_t = \beta_0 + \beta_1 x_{t1} + \beta_2 x_{t2} + \beta_3 x_{t3} + \beta_4 x_{t4} + \beta_5 x_{t5},$$

for $t = 1, \dots, 34$. The covariates are the scores in the following events: javelin throw (x_1), long jump (x_2), discus throw (x_3), shot put (x_4), and pole vault (x_5). The statistics and the corresponding p -values for testing $\mathcal{H}_0 : \beta_1 = 0$ against $\mathcal{H}_1 : \beta_1 \neq 0$ are presented in Table 5.

Note that the p -values vary from 0.0168 (Wald test) to 0.1038 (adjusted likelihood ratio test). Only the likelihood ratio and Wald tests reject the null hypothesis at the 5% nominal level, but this conclusion may be misleading since these tests showed liberal behaviour in our simulations. The most reliable test, namely, the adjusted likelihood ratio test, does not suggest that the null hypothesis should be rejected (p -value = 0.1038).

²The dataset is also available at <http://www.stat.ncsu.edu/working-groups/sas/sicl/data/olympic.dat>.

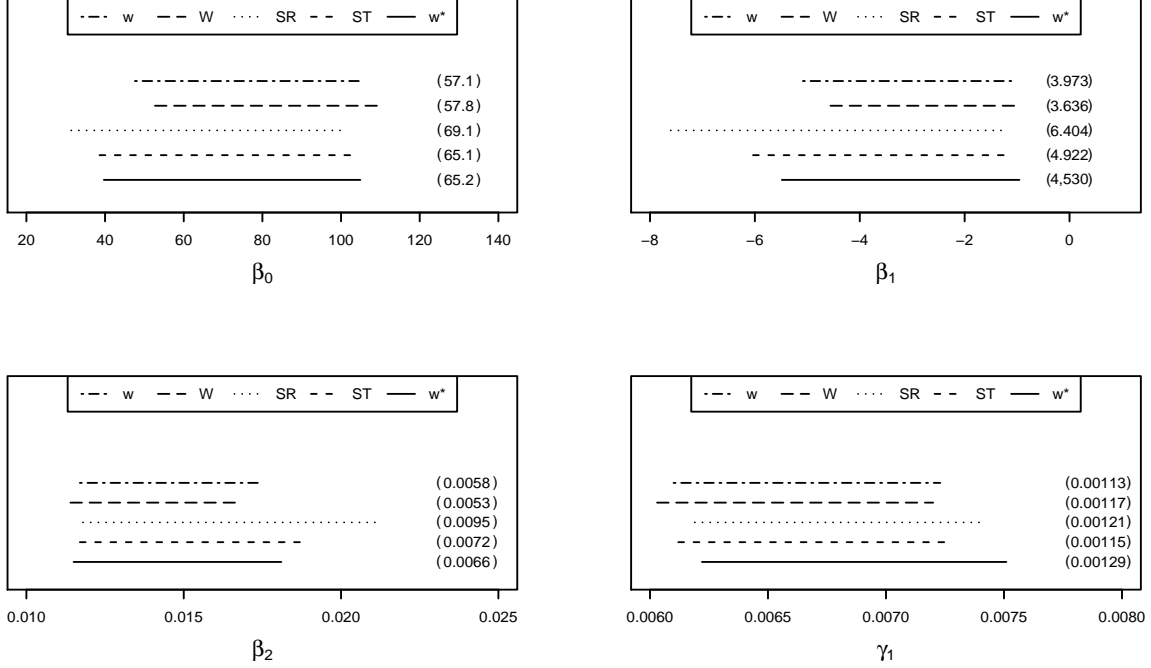


Figure 5: 95% confidence intervals; length in parentheses.

Table 5: Test statistics and the corresponding p -values

	w	W	S_R	S_T	w^*
statistic	4.0407	5.7161	2.8208	3.6293	2.6466
p -value	0.0444	0.0168	0.0930	0.0568	0.1038

6 Conclusion

In this paper, we derived an adjusted version of the likelihood ratio statistic that provides accurate inference in extreme-value regression models in small- to moderate-sized samples. Our simulation results suggest that the likelihood ratio and the Wald tests can be markedly oversized in small- and moderate-sized samples. The gradient test can be oversized, but much less than the other two tests. The score test is even less size distorted and can be conservative in some cases. We emphasize that the score test performs clearly better than the likelihood ratio, Wald and gradient tests, and is competitive with the adjusted likelihood ratio test in most cases. The adjusted likelihood ratio test obtained in this paper performs better than all the others. Although it requires more computational effort, is the least size distorted in most cases and it is, therefore, recommended for practical applications. We emphasize that our simulations were carried out in extreme-value regression models with linear and non-linear predictors for

both location and dispersion parameters. All simulation results exhibited reasonably similar behavior.

Acknowledgements

We thank the referee for helpful comments and suggestions. We gratefully acknowledge the financial support from CNPq.

Appendix

Let $y \sim EV_{max}(\mu, \phi)$ and $z = (y - \mu)/\phi \sim EV_{max}(0, 1)$. We have

$$E(z^n \exp(-cz)) = \int_{-\infty}^{\infty} z^n \exp(-cz) \exp(-z - \exp(-z)) dz, \quad n = 0, 1, \dots$$

Using the transformation $v = \exp(-z)$, we have

$$\begin{aligned} E(z^n \exp(-cz)) &= \int_0^1 (-\ln v)^n v^c v \exp(-v) \frac{-1}{v} dv = \int_0^1 (-1)^n (\ln v)^n v^c \exp(-v) dv \\ &= (-1)^n \Gamma^{(n)}(1+c), \quad n = 1, 2, \dots; \end{aligned}$$

see (4.358) in Gradshteyn et al. (2000). Since $\Gamma^{(1)}(n) = -(n-1)!(1/n + \mathcal{E} - \sum_{k=1}^n 1/k)$, we obtain

$$\begin{aligned} E(z) &= \mathcal{E}, \quad E(z^2) = \Gamma^{(2)}(1), \quad E(\exp(-cz)) = \Gamma(1+c), \quad E(\exp(-(1+c)z)) = \Gamma(2+c), \\ E(z \exp(-z)) &= \mathcal{E} - 1, \quad E(z \exp(-2z)) = 2\mathcal{E} - 3, \quad E(z \exp(-cz)) = -\Gamma^{(1)}(1+c), \\ E(z \exp(-(1+c)z)) &= -\Gamma^{(1)}(2+c), \quad E(z^2 \exp(-z)) = \Gamma^{(2)}(2), \\ E(z^2 \exp(-cz)) &= \Gamma^{(2)}(1+c), \quad E(z^2 \exp(-(1+c)z)) = \Gamma^{(2)}(2+c). \end{aligned}$$

Let $e = (\exp(-(\phi_{11}/\phi_1)\mathfrak{z}_1), \dots, \exp(-(\phi_{1n}/\phi_n)\mathfrak{z}_n))^T$. Since $\Gamma^{(1)}(n) = \Gamma(n-1) + (n-1)\Gamma^{(1)}(n-1)$ and $\Gamma^{(2)}(n) = 2\Gamma^{(1)}(n-1) + (n-1)\Gamma^{(2)}(n-1)$ we can write after some calculations that

$$\begin{aligned} E_\omega(\check{\mathfrak{z}}) &= \mathbf{1}, \quad E_\omega(\mathfrak{z}) = \mathcal{E}\mathbf{1}, \quad E_\omega(\mathcal{Z}) = \mathcal{E}\mathcal{I}, \quad E_\omega(\check{\mathcal{Z}}) = \mathcal{I}, \quad E_\omega(\mathcal{Z}\check{\mathfrak{z}}) = (\mathcal{E}-1)\mathbf{1}, \\ E_\omega(\mathcal{Z}\check{\mathcal{Z}}) &= (\mathcal{E}-1)\mathcal{I}, \quad E_\omega(\mathcal{Z}^2\check{\mathcal{Z}}) = \Gamma^{(2)}(2)\mathcal{I}, \quad E_\omega(e) = M\mathbf{1}, \quad E_\omega(\check{\mathfrak{z}}\check{\mathfrak{z}}^\top) = \mathbf{1}\mathbf{1}^\top + \mathcal{I}, \\ E_\omega(\check{\mathfrak{z}}e^\top) &= \mathbf{1}\mathbf{1}^\top M + CM, \quad E_\omega(\check{\mathfrak{z}}\check{\mathfrak{z}}^\top) = \mathcal{E}\mathbf{1}\mathbf{1}^\top - \mathcal{I}, \quad E_\omega(\check{\mathfrak{z}}\check{\mathfrak{z}}^\top) = E_\omega(\check{\mathfrak{z}}\check{\mathfrak{z}}^\top), \\ E_\omega(\mathfrak{z}e^\top) &= \mathcal{E}\mathbf{1}\mathbf{1}^\top M - \mathcal{E}M - N, \quad E_\omega(\mathfrak{z}\check{\mathfrak{z}}^\top) = \mathcal{E}^2\mathbf{1}\mathbf{1}^\top - \mathcal{E}^2\mathbf{I} + \Gamma^{(2)}(1)\mathcal{I}, \quad E_\omega(\mathcal{Z}e) = -N\mathbf{1}, \\ E_\omega(\check{\mathfrak{z}}e^\top \mathcal{Z}) &= -\mathbf{1}\mathbf{1}^\top N - M - CN, \quad E_\omega(\mathfrak{z}e^\top \mathcal{Z}) = -\mathcal{E}\mathbf{1}\mathbf{1}^\top N + \mathcal{E}N + P, \\ E_\omega(\mathcal{Z}\check{\mathfrak{z}}\check{\mathfrak{z}}^\top) &= (\mathcal{E}-1)\mathbf{1}\mathbf{1}^\top + (\mathcal{E}-2)\mathcal{I}, \\ E_\omega(\mathcal{Z}\check{\mathfrak{z}}e^\top) &= (\mathcal{E}-1)\mathbf{1}\mathbf{1}^\top M - (\mathcal{E}-1)M - M - (\mathcal{I}+C)N, \\ E_\omega(\mathcal{Z}\check{\mathfrak{z}}\check{\mathfrak{z}}^\top) &= \mathcal{E}(\mathcal{E}-1)\mathbf{1}\mathbf{1}^\top - \mathcal{E}(\mathcal{E}-1)\mathcal{I} + \Gamma^{(2)}(2)\mathcal{I}, \\ E_\omega(\mathcal{Z}\check{\mathfrak{z}}e^\top \mathcal{Z}) &= -(\mathcal{E}-1)\mathbf{1}\mathbf{1}^\top N + (\mathcal{E}-1)N + 2N + P + CP. \end{aligned}$$

Now, let $y_t \sim EV_{max}(\mu_{1t}, \phi_{1t})$, $\mathbf{z}_{1t} = (y_t - \mu_{1t})/\phi_{1t} \sim EV_{max}(0, 1)$ and $\mathbf{z}_t = (y_t - \mu_t)/\phi_t$. We can write $\mathbf{z}_t = (\phi_{1t}/\phi_t)\mathbf{z}_{1t} + (\mu_{1t} - \mu_t)/\phi_t$. Therefore,

$$\mathbf{z} = C\mathbf{z}_1 + D\mathbf{1} \quad (12)$$

and

$$\check{\mathbf{z}} = \check{D}e_1, \quad (13)$$

where $e_1 = (\exp(-(\phi_{11}/\phi_1)\mathbf{z}_{11}), \dots, \exp(-(\phi_{1n}/\phi_n)\mathbf{z}_{1n}))^\top$ and the other quantities are as given earlier.

The vector q given in (8) is given by

$$q = \begin{bmatrix} E_{\omega_1}[U_\beta(\omega_1)\ell(\omega_1)] - E_{\omega_1}[U_\beta(\omega_1)\ell(\omega)] \\ E_{\omega_1}[U_\gamma(\omega_1)\ell(\omega_1)] - E_{\omega_1}[U_\gamma(\omega_1)\ell(\omega)] \end{bmatrix}.$$

From (4), (5), and the expected values obtained above, we have

$$\begin{aligned} E_{\omega_1}[U_\beta(\omega_1)\ell(\omega_1)] &= E_{\omega_1}\{X_1^\top \Phi_1^{-1} T_1 (\mathbf{1} - \check{\mathbf{z}}_1) [-l^\top - \check{\mathbf{z}}_1^\top - \check{\mathbf{z}}_1^\top] \mathbf{1}\} \\ &= X_1^\top \Phi_1^{-1} T_1 \{-\mathbf{1}l^\top - \mathbf{1}E_{\omega_1}(\check{\mathbf{z}}_1^\top) - \mathbf{1}E_{\omega_1}(\check{\mathbf{z}}_1^\top) + E_{\omega_1}(\check{\mathbf{z}}_1)l^\top + E_{\omega_1}(\check{\mathbf{z}}_1\check{\mathbf{z}}_1^\top) + E_{\omega_1}(\check{\mathbf{z}}_1\check{\mathbf{z}}_1^\top)\} \mathbf{1} \\ &= X_1^\top \Phi_1^{-1} T_1 \{-\mathbf{1}l^\top - \mathcal{E}\mathbf{1}\mathbf{1}^\top - \mathbf{1}\mathbf{1}^\top + \mathbf{1}l^\top + (\mathcal{E}\mathbf{1}\mathbf{1}^\top - \mathcal{I}) + (\mathbf{1}\mathbf{1}^\top + \mathcal{I})\} \mathbf{1} = \mathbf{0}\mathbf{1}. \end{aligned}$$

Now, from (12) and (13), we have

$$\begin{aligned} E_{\omega_1}[U_\beta(\omega_1)\ell(\omega)] &= E_{\omega_1}\{X_1^\top \Phi_1^{-1} T_1 (\mathbf{1} - \check{\mathbf{z}}_1) [-l^\top - \check{\mathbf{z}}_1^\top - \check{\mathbf{z}}_1^\top] \mathbf{1}\} \\ &= X_1^\top \Phi_1^{-1} T_1 E_{\omega_1}\{(\mathbf{1} - \check{\mathbf{z}}_1) [-l^\top - \check{\mathbf{z}}_1^\top C - \mathbf{1}^\top D - e_1^\top \check{D}] \mathbf{1}\} \\ &= X_1^\top \Phi_1^{-1} T_1 \{-\mathbf{1}l^\top - \mathbf{1}E_{\omega_1}(\check{\mathbf{z}}_1^\top)C - \mathbf{1}\mathbf{1}^\top D - \mathbf{1}E_{\omega_1}(e_1^\top)\check{D} + E_{\omega_1}(\check{\mathbf{z}}_1)l^\top + E_{\omega_1}(\check{\mathbf{z}}_1\check{\mathbf{z}}_1^\top)C + \\ &\quad E_{\omega_1}(\check{\mathbf{z}}_1)\mathbf{1}^\top D + E_{\omega_1}(\check{\mathbf{z}}_1e_1^\top)\check{D}\} \mathbf{1} \\ &= X_1^\top \Phi_1^{-1} T_1 \{-\mathbf{1}l^\top - \mathcal{E}\mathbf{1}\mathbf{1}^\top C - \mathbf{1}\mathbf{1}^\top D - \mathbf{1}(\mathbf{1}^\top M)\check{D} + \mathbf{1}l^\top + (\mathcal{E}\mathbf{1}\mathbf{1}^\top - \mathcal{I})C + \\ &\quad \mathbf{1}\mathbf{1}^\top D + (\mathbf{1}\mathbf{1}^\top M + CM)\check{D}\} \mathbf{1} = X_1^\top \Phi_1^{-1} T_1 C(M\check{D} - \mathcal{I})\mathbf{1}. \end{aligned}$$

Hence,

$$E_{\omega_1}[U_\beta(\omega_1)\ell(\omega_1)] - E_{\omega_1}[U_\beta(\omega_1)\ell(\omega)] = X_1^\top \Phi_1^{-1} T_1 C(\mathcal{I} - M\check{D})\mathbf{1}.$$

From (4), (6), and the results above, we have

$$\begin{aligned} E_{\omega_1}[U_\gamma(\omega_1)\ell(\omega_1)] &= E_{\omega_1}\{Z_1^\top \Phi_1^{-1} H_1 (\mathbf{z}_1 - \mathcal{Z}_1\check{\mathbf{z}}_1 - \mathbf{1}) [-l^\top - \check{\mathbf{z}}_1^\top - \check{\mathbf{z}}_1^\top] \mathbf{1}\} \\ &= Z_1^\top \Phi_1^{-1} H_1 \{-E_{\omega_1}(\mathbf{z}_1)l^\top - E_{\omega_1}(\mathbf{z}_1\check{\mathbf{z}}_1^\top) - E_{\omega_1}(\mathbf{z}_1\check{\mathbf{z}}_1^\top) + E_{\omega_1}(\mathcal{Z}_1\check{\mathbf{z}}_1)l^\top \\ &\quad + E_{\omega_1}(\mathcal{Z}_1\check{\mathbf{z}}_1\check{\mathbf{z}}_1^\top) + E_{\omega_1}(\mathcal{Z}_1\check{\mathbf{z}}_1\check{\mathbf{z}}_1^\top) + \mathbf{1}l^\top + \mathbf{1}E_{\omega_1}(\check{\mathbf{z}}_1^\top) + \mathbf{1}E_{\omega_1}(\check{\mathbf{z}}_1^\top)\} \mathbf{1} \\ &= Z_1^\top \Phi_1^{-1} H_1 \{-\Gamma^{(2)}(1)\mathcal{I} + 2\mathcal{E}\mathcal{I} + \Gamma^{(2)}(2)\mathcal{I} - \mathcal{I}\} \mathbf{1} \\ &= -Z_1^\top \Phi_1^{-1} H_1 \mathbf{1}, \end{aligned}$$

and from (12) and (13) we get

$$\begin{aligned}
\mathbb{E}_{\omega_1}[U_\gamma(\omega_1)l(\omega)] &= \mathbb{E}_{\omega_1}\{Z_1^\top \Phi_1^{-1} H_1(\mathfrak{z}_1 - \mathcal{Z}_1 \check{\mathfrak{z}}_1 - \mathbf{1})[-l^\top - \check{\mathfrak{z}}_1^\top - \check{\mathfrak{z}}_1^\top] \mathbf{1}\} \\
&= \mathbb{E}_{\omega_1}\{Z_1^\top \Phi_1^{-1} H_1(\mathfrak{z}_1 - \mathcal{Z}_1 \check{\mathfrak{z}}_1 - \mathbf{1})[-l^\top - \check{\mathfrak{z}}_1^\top C - \mathbf{1}^\top D - e_1^\top \check{D}] \mathbf{1}\} \\
&= Z_1^\top \Phi_1^{-1} H_1 \{-\mathbb{E}_{\omega_1}(\mathfrak{z}_1) l^\top - \mathbb{E}_{\omega_1}(\mathfrak{z}_1 \check{\mathfrak{z}}_1^\top) C - \mathbb{E}_{\omega_1}(\mathfrak{z}_1) \mathbf{1}^\top D - \mathbb{E}_{\omega_1}(\mathfrak{z}_1 e_1^\top) \check{D} \\
&\quad + \mathbb{E}_{\omega_1}(\mathcal{Z}_1 \check{\mathfrak{z}}_1) l^\top + \mathbb{E}_{\omega_1}(\mathcal{Z}_1 \check{\mathfrak{z}}_1 \check{\mathfrak{z}}_1^\top) C + \mathbb{E}_{\omega_1}(\mathcal{Z}_1 \check{\mathfrak{z}}_1) \mathbf{1}^\top D + \mathbb{E}_{\omega_1}(\mathcal{Z}_1 \check{\mathfrak{z}}_1 e_1^\top) \check{D} \\
&\quad + \mathbf{1} l^\top + \mathbf{1} \mathbb{E}_{\omega_1}(\check{\mathfrak{z}}_1^\top) C + \mathbf{1} \mathbf{1}^\top D + \mathbf{1} \mathbb{E}_{\omega_1}(e_1^\top) \check{D}\} \mathbf{1} \\
&= -Z_1^\top \Phi_1^{-1} H_1 C \{\mathcal{E} \mathcal{I} + N \check{D}\} \mathbf{1}.
\end{aligned}$$

It follows that

$$\mathbb{E}_{\omega_1}[U_\gamma(\omega_1)l(\omega_1)] - \mathbb{E}_{\omega_1}[U_\gamma(\omega_1)l(\omega)] = Z_1^\top \Phi_1^{-1} H_1 (C(\mathcal{E} \mathcal{I} + N \check{D}) - \mathcal{I}) \mathbf{1}.$$

Hence,

$$q = \begin{bmatrix} X_1^\top \Phi_1^{-1} T_1 C(\mathcal{I} - M \check{D}) \mathbf{1} \\ Z_1^\top \Phi_1^{-1} H_1 \{C(\mathcal{E} \mathcal{I} + N \check{D}) - \mathcal{I}\} \mathbf{1} \end{bmatrix}.$$

The matrix Υ given in (9) can be written as

$$\Upsilon = \begin{bmatrix} \mathbb{E}_{\omega_1}[U_\beta(\omega_1)U_\beta^\top(\omega)] & \mathbb{E}_{\omega_1}[U_\beta(\omega_1)U_\gamma(\omega)] \\ \mathbb{E}_{\omega_1}[U_\gamma(\omega_1)U_\beta^\top(\omega)] & \mathbb{E}_{\omega_1}[U_\gamma(\omega_1)U_\gamma^\top(\omega)] \end{bmatrix}.$$

From (5), (13), and the expected values obtained in the beginning of this Appendix, we have

$$\begin{aligned}
\mathbb{E}_{\omega_1}\{U_\beta(\omega_1)U_\beta^\top(\omega)\} &= \mathbb{E}_{\omega_1}\{X_1^\top \Phi_1^{-1} T_1 (\mathbf{1} - \check{\mathfrak{z}}_1) [X^\top \Phi^{-1} T (\mathbf{1} - \check{D} e_1)]^\top\} \\
&= X_1^\top \Phi_1^{-1} T_1 (\mathbf{1} \mathbf{1}^\top - \mathbf{1} \mathbb{E}_{\omega_1}(e_1^\top) \check{D} - \mathbb{E}_{\omega_1}(\check{\mathfrak{z}}_1) \mathbf{1}^\top + \mathbb{E}_{\omega_1}(\check{\mathfrak{z}}_1 e_1^\top) \check{D}) T \Phi^{-1} X \\
&= X_1^\top \Phi_1^{-1} T_1 (\mathbf{1} \mathbf{1}^\top - \mathbf{1} \mathbf{1}^\top M \check{D} - \mathbf{1} \mathbf{1}^\top + (\mathbf{1} \mathbf{1}^\top M + C M) \check{D}) T \Phi^{-1} X \\
&= X_1^\top \Phi_1^{-1} T_1 C M \check{D} T \Phi^{-1} X.
\end{aligned}$$

The other blocks of Υ are derived in a similar fashion. We obtained

$$\mathbb{E}_{\omega_1}\{U_\beta(\omega_1)U_\gamma^\top(\omega)\} = X_1^\top \Phi_1^{-1} T_1 C \{\mathcal{I} + \check{D}(M D - M - C N)\} H \Phi^{-1} Z,$$

$$\mathbb{E}_{\omega_1}\{U_\gamma(\omega_1)U_\beta^\top(\omega)\} = -Z_1^\top \Phi_1^{-1} H_1 C N \check{D} T \Phi^{-1} X,$$

$$\mathbb{E}_{\omega_1}\{U_\gamma(\omega_1)U_\gamma^\top(\omega)\} = Z_1^\top \Phi_1^{-1} H_1 C \{\mathcal{E} \mathcal{I} + \check{D}(N + C P - N D)\} H \Phi^{-1} Z.$$

Therefore,

$$\Upsilon = \begin{bmatrix} X_1^\top \Phi_1^{-1} T_1 C M \check{D} T \Phi^{-1} X & X_1^\top \Phi_1^{-1} T_1 C \{\mathcal{I} + \check{D}(M D - M - C N)\} H \Phi^{-1} Z \\ -Z_1^\top \Phi_1^{-1} H_1 C N \check{D} T \Phi^{-1} X & Z_1^\top \Phi_1^{-1} H_1 C \{\mathcal{E} \mathcal{I} + \check{D}(N + C P - N D)\} H \Phi^{-1} Z \end{bmatrix}.$$

The vector \bar{q} and the matrix $\bar{\Upsilon}$ are obtained from q and Υ given above by replacing X_1 , Φ_1 , T_1 , Z_1 , and H_1 by \hat{X} , $\hat{\Phi}$, \hat{T} , \hat{Z} , and \hat{H} , respectively, and X , Φ , T , Z , and H by \tilde{X} , $\tilde{\Phi}$, \tilde{T} , \tilde{Z} , and \tilde{H} , respectively.

References

- Barreto-Souza, W., Vasconcellos, K.L.P. (2011). Bias and skewness in a general extreme-value regression model. *Computational Statistics & Data Analysis*, **55**, 1379–1393.
- Castillo, E., Hadi, A.S., Balakrishnan, N., Sarabia, J.M. (2005). *Extreme Value and Related Models with Applications in Engineering and Science*. New Jersey: Wiley.
- Coles, S. (2001). *An Introduction to Statistical Modeling of Extremes*. London: Springer-Verlag.
- Doornik, J.A. (2009). *Ox 6 - An Object-Oriented Matrix Language*. London: Timberlake Consultants Press.
- Faivre, R., Masle, J. (1988). Modeling potential growth of tillers in winter wheat. *Acta Ecologica*, **9**, 179–196.
- Ferrari, S.L.P., Cysneiros, A.H.M.A. (2008) Skovgaard’s adjustment to likelihood ratio tests in exponential family nonlinear models. *Statistics and Probability Letters*, **78**, 3047–3055.
- Ferrari, S.L.P., Pinheiro, E.C. (2010). Improved likelihood inference in beta regression. *Journal of Statistical Computation and Simulation*, **81**, 431–443.
- Gradshteyn, I.S., Ryzhik, I.M. (2000). *Table of Integrals, Series and Products*. Massachusetts: Academic Press.
- Gumbel, E.J. (1958). *Statistics of Extremes*. 2nd ed. New York: Columbia University Press.
- Hand, D.J., Daly, F., Lunn, A.D., McConway, K.J., Ostrowsky, E. (1996). *A Handbook of Small Data Sets*. 2nd ed. London: Chapman and Hall.
- Huet, S., Bouvier, A., Poursat, M.-A., Jolivet, E. (2004). *Statistical Tools for Nonlinear Regression*. 2nd ed. New York: Springer-Verlag.
- Kotz, S., Nadarajah, S. (2000). *Extreme Value Distributions: Theory and Applications*. London: Imperial College Press.
- Mueller, S., Rufibach, K. (2009). Smooth tail-index estimation. *Journal of Statistical Computation and Simulation*, **79**, 1155–1167.
- Press, W.H., Teulosky, S.A., Vetterling, W.T., Flannery, B.P. (1992). *Numerical Recipes in C: The Art of Scientific Computing*. 2nd ed. London: Prentice Hall.
- Skovgaard, I.M. (2001). Likelihood asymptotics. *Scandinavian Journal of Statistics*, **28**, 3–32.
- Terrell, G.R. (2002). The gradient statistic. *Computing Science and Statistics*, **34**, 206–215.